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Towards the Ultimate Conservative Difference Scheme III. Upstream-Centered Finite-Difference Schemes for Ideal Compressible Flow

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Finite-difference schemes for the conservation laws of ideal compressible flow are constructed on the basis of upstream-centered convective schemes, Fromm's second-order scheme in particular. The upstream centering generates a number of higher-order terms, making the schemes quite complex. In consequence, they seem to compare unfavorably with central-difference schemes as regards computational efficiency. Previously derived upstream-centered terms that prevent numerical oscillations in Fromm's scheme partly lose their effect when included in a version of the scheme for compressible flow. Apparently, the finite-difference approach is of little avail in formulating upstream schemes for compressible flow. It is anticipated that Godunov's approach, involving more of the physics in the conservation laws, will lead to more attractive schemes.

1. INTRODUCTION

In constructing finite-difference schemes for the convection equation $\omega_t + a\omega_x = 0$, a distinction can be made between "central differencing" and "upstream differencing".

In a central-difference scheme, the numerical domain of dependence of the convected quantity ω is centered on the point in space, say x_0, where the value of ω is being updated. An example is the second-order scheme of Lax and Wendroff (LW) [1], well known for the numerous ways in which it has been applied to the conservation laws of ideal compressible flow (ICF).

In an upstream-difference scheme, the numerical domain of dependence is centered on the point $x_0 - a\,\Delta t$, from where the convection path departed in the past. Examples are the first-order scheme of Courant, Isaacson and Rees (CIR) [2] for the characteristic equations of ICF, and the second-order scheme of Fromm [3] for the convection of vorticity. The CIR scheme has been put into the conservation form by Godunov [4] in a way that deviates from the common finite-difference approach. In particular, the Godunov version involves a great deal of the physical content of the conservation laws. No conservative version for ICF exists of any other upstream-centered convective scheme. This is a real lack, since upstream schemes are known for their low phase errors.

In the present paper I shall show how such upstream conservative schemes can

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be constructed following the finite-difference approach. The procedure applies to the basic convective schemes as well as to the nonlinear feedback terms, derived in the previous paper [19], that are needed to prevent numerical oscillations. The resulting schemes, however, are computationally not very attractive when compared to central-difference schemes of about the same accuracy.

Fortunately it is possible, by following Godunov's approach, to formulate upstream conservative schemes that are more appealing. Since the latter approach is conceptually different from the former and also calls for a different notation, its discussion will be deferred to future installments of the present series.

2. The First-Order Upstream Conservative Scheme

The favorable properties of upstream convective schemes are well documented (see, e.g., [5]). In such schemes, the dispersive and dissipative errors are more closely balanced than in schemes of equal or even higher order that use one and the same set of nodal points, regardless of the direction of convection. The advantage shows up notably well when the initial-value distribution of the convected quantity contains strong shortwave components. Upstream convective schemes, therefore, are pre-eminently suited as a starting point in designing shock-handling schemes for ICF.

It appears, however, that upstream schemes are more often praised than used. Godunov's Lagrangean scheme is described by Richtmyer and Morton [6] as "an ingeneous method" and is shown to yield a much better shock profile than the popular first-order scheme of Von Neumann and Richtmyer [7]. The results of the Euler version [8] compare favorably to those obtained with Rusanov's [9] first-order scheme (see [10]). Yet, Godunov's scheme is hardly used outside the Soviet Union.

The obvious reason for its obscurity is its computational complexity, which also mystifies the relation of the scheme to other conservative schemes, such as Lax and Wendroff's [1]. In Van Leer [11] this relation is clarified and a two-step version of the scheme is constructed in the manner Burstein and Rubin [12] indicated for the LW scheme. I shall repeat the argument further down in this section.

Fromm's scheme, formulated only for a single convection equation, has also been highly acclaimed (see, e.g., [13]) but appears to be used mainly by its inventor. Likewise, it has been the subject of mystification, caused by the peculiar way the scheme was originally derived. From [3] it is not at all clear that Fromm's "zero-average phase-error method" is just the simplest upstream second-order scheme. The resulting confusion may be illustrated by quoting Roache [14, p. 105], who suggests the following "Exercise: Design a method of zero-average phase error [by folowing Fromm's line of reasoning] based on upwind differencing [that is, on the CIR scheme]." The outcome is that the CIR scheme remains unchanged, being a "zero-average phase-error method" itself!

The cases of mystification reported above are not accidental; upstream differencing requires a deeper involvement than central differencing, especially when applied to a nonlinear system of equations in conservation form. We shall see this below.

Consider the hyperbolic system of nonlinear conservation laws

$$(\partial w/\partial t) + (\partial f/\partial x) = 0, \tag{1}$$

representing, e.g., the one-dimensional Lagrange or Euler equations of ICF, or a one-dimensional fraction of the many-dimensional Euler equations, used in a time-splitting procedure. Defining the matrix

$$A \equiv df/dw, \tag{2}$$

we can write (1) as

$$(\partial w/\partial t) + A(\partial w/\partial x) = 0, \tag{3}$$

which, through left multiplication by a suitable matrix, reduces to the normal form

$$(\partial \omega/\partial t) + \mathscr{A}(\partial \omega/\partial x) = 0. \tag{4}$$

Here $\mathscr{A}$ is a diagonal matrix; its diagonal elements are the eigenvalues $a^{(k)}$ of A and are called the characteristic speeds. These are all real, but need not all be distinct. The new state vector ω must in general be regarded as a function of x and t, not of w alone.

Along the kth characteristic trajectory some quantity is conserved, according to the nonlinear convection equation

$$(\partial \omega^{(k)}/\partial t) + a^{(k)}(\partial \omega^{(k)}/\partial x) = 0. \tag{5}$$

In order to arrive at a suitable starting point for designing difference schemes, I shall assume that $a^{(k)}$ is a constant and drop the superscripts:

$$(\partial \omega/\partial t) + a(\partial \omega/\partial x) = 0. \tag{6}$$

Now define a uniform computational net $\{x_i, t^n\}$, with mesh sizes Δx and Δt constrained by some stability condition. For all schemes considered below this is the usual Courant–Friedrichs–Lewy (CFL) condition

$$(\Delta t/\Delta x) \mid a \mid \leqslant 1. \tag{7}$$

It is convenient to introduce

$$\lambda \equiv \Delta t/\Delta x, \tag{8}$$

the mesh ratio, and

$$\sigma \equiv (\Delta t/\Delta x)\, a, \tag{9}$$

a quantity of which the absolute value is called the Courant number. The complete notation is compiled in Table I. The time index has been suppressed; subscripts denote initial values and superscripts denote final values. Otherwise the notation is trivial.

TABLE I

Notation Used in the Grid

Symbol	Definition
x_0	Abcissa where the time difference of w is evaluated
x_i	$x_0 + i\Delta x$
t_0	Initial time level
t'	$t_0 + \Delta t$, final time level
w_i	$w(t^0, x_i)$, initial value of w in x_i
w^0	$w(t^1, x_0)$, final value of w in x_0
$\Delta_{i+1/2}w$	$w_{i+1} - w_i$
$w_{i+1/2}$	$\frac{1}{2}(w_i + w_{i+1})$
$\Delta_i w$	$w_{i+1/2} - w_{i-1/2}$
λ	$\Delta t/\Delta x$, mesh ratio
σ	λa, signed Courant number

The scheme used by CIR for the characteristic equations (4) takes the following form when applied to Eq. (6):

$$w^0 = \begin{cases} w_0 - \sigma\Delta_{-1/2}w, & \text{if } \sigma \geqslant 0, \\ w_0 - \sigma\Delta_{1/2}w, & \text{if } \sigma < 0. \end{cases} \tag{10}$$

In words, w^0 is the result of linear interpolation in $x_0 - a\,\Delta t$ between the initial values of w in the adjacent nodal points. Scheme (10) therefore switches between using w_{-1} and using w_1, whichever is the upstream value. When transforming the scheme back into a conservative scheme for Eq. (1), we shall need local knowledge of the signs of the characteristic speeds. A conventional way to go about the transformation is given next.

First, rewrite scheme (10) using the linear operators L_+ and L_-:

$$w^0 = L_+ w_0 \qquad \text{for} \quad \sigma = |\,\sigma\,|, \tag{11a}$$

$$w^0 = L_- w_0 \qquad \text{for} \quad \sigma = -\,|\,\sigma\,|. \tag{11b}$$

Next, combine (11a) and (11b) into

$$w^0 = \left(\frac{L_+ + L_-}{2} + \frac{|\,\sigma\,|}{\sigma}\,\frac{L_+ - L_-}{2}\right) w_0, \tag{12}$$

which in finite-difference notation reads

$$w^0 = w_0 - (\sigma/2)(\Delta_{1/2}w + \Delta_{-1/2}w) + (|\sigma|/2)(\Delta_{1/2}w - \Delta_{-1/2}w). \tag{13}$$

Thus, switching is done by the modulus bars in the coefficient of the second-order term.

Now introduce a matrix $\tilde{A}$, commuting with A, with eigenvalues that are the *absolute values* of the corresponding eigenvalues of A. Symbolically,

$$\tilde{A} \equiv (A^2)^{1/2}. \tag{14}$$

Using this definition we can derive from (13) the following conservative scheme for Eq. (1):

$$w^0 = w_0 - \lambda\, \Delta_0 f + (\lambda/2)(\tilde{A}_{1/2}\, \Delta_{1/2} w - \tilde{A}_{-1/2}\, \Delta_{-1/2} w). \tag{15}$$

This scheme differs from the actual Godunov scheme only by terms of the magnitude $O[(\Delta x)^2\, \Delta A]$.

Scheme (15) invites comparison with the second-order accurate scheme of Lax and Wendroff [1], which reads

$$w^0 = w_0 - \lambda\, \Delta_0 f + (\lambda^2/2)(A^2_{1/2}\, \Delta_{1/2} w - A^2_{-1/2}\, \Delta_{-1/2} w). \tag{16}$$

Both (15) and (16) are members of the family of Q schemes

$$w^0 = w_0 - \lambda\, \Delta_0 f + \tfrac{1}{2}(Q_{1/2}\, \Delta_{1/2} w - Q_{-1/2}\, \Delta_{-1/2} w), \tag{17}$$

with

$$Q_{\mathrm{Godunov}} = \lambda \tilde{A} \tag{18}$$

and

$$Q_{\mathrm{LW}} = \lambda^2 A^2. \tag{19}$$

The family as a whole was described by Van Leer [11]. The most useful schemes result when Q is chosen to commute with A; this will be assumed henceforth. From [11] I repeat that Q schemes are stable provided

$$\lambda^2 A^2 \leqslant Q \leqslant I, \tag{20}$$

and that they satisfy the linear monotonicity condition if

$$\lambda \tilde{A} \leqslant Q \leqslant I. \tag{21}$$

Scheme (15) appears to be a "best buy," representing a trade-off between accuracy and smoothness of results. This is the very reason why Godunov adopted it. A more general formulation of the same property is given by Wesseling [5]. Figure 1 shows the different performance of schemes (15) and (16), when applied to the same shocked-flow problem.

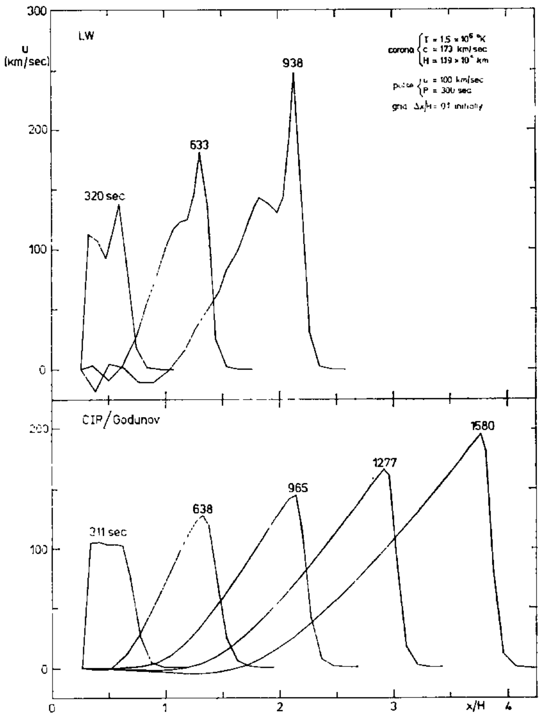

FIG. 1. Propagation of a shock wave through the solar corona. A convective cell at the solar surface moves outward at a velocity of 100 km/sec for a period of 300 sec. A shock wave forms, accelerating as it meets smaller densities and pressures higher up in the isothermal atmosphere. Top: results of Lagrangean version of scheme (16). Bottom: results of Lagrangean version of scheme (15). Courtesy of R. van Hees, University Observatory, Utrecht, The Netherlands.

The vector-by-matrix multiplications $\tilde{A}\,\Delta w$ and $A^2\,\Delta w$ (or $A\,\Delta f$) occurring in (15) and (16) make these schemes inefficient, at least for the Euler equations (see [15]). For the LW scheme, various two-step formulations have been introduced that avoid the multiplication; one form, given by Burstein and Rubin [12], is

$$
\begin{aligned}
w^*_{1/2} &= w_{1/2} - (\lambda/2)\,\Delta_{1/2} f,\\
w^0 &= w_0 - \lambda\,\Delta_0 f^*.
\end{aligned}
\qquad (22)
$$

This deviates a mere $O[(\Delta x)^2 \Delta A]$ from the original LW scheme, since

$$f^*_{1/2} = f_{1/2} - (\lambda/2) A_{1/2} \Delta_{1/2} f + O(\Delta x \Delta A). \tag{23}$$

Any Q scheme for ICF allows a two-step formulation in which the multiplication $Q \Delta w$ is avoided. For ICF, the matrix A always has three distinct eigenvalues; therefore, Q can be uniquely written as a quadratic polynomial in λA (see [1, p. 228] or [11, p. 22]):

$$Q = q_0 I + q_1 \lambda A + q_2 \lambda^2 A^2. \tag{24}$$

This expansion allows the following two-step formulation of the general Q scheme (17):

$$\begin{aligned} w^*_{1/2} &= w_{1/2} - (\lambda/2)(q_2)_{1/2} \Delta_{1/2} f, \\ w^0 &= w_0 - \lambda \Delta_0 f^* + \tfrac{1}{2}\{(q_0)_{1/2} \Delta_{1/2} w - (q_0)_{-1/2} \Delta_{-1/2} w\} \\ &\quad + (\lambda/2)\{(q_1)_{1/2} \Delta_{1/2} f - (q_1)_{-1/2} \Delta_{-1/2} f\}. \end{aligned} \tag{25}$$

The q_1 term may also be incorporated in the first step as $-\frac{1}{2}(q_1)_{1/2} \Delta_{1/2} w$, or be divided over the two steps. See also Richtmyer and Morton [6, p. 335ff], where the same technique is indicated for the inclusion of extra dissipation in the LW scheme.

To demonstrate the use of expansion (24) in scheme (15), consider the Lagrangean equations. These imply the characteristic speeds $-C$, 0 and $+C$, where C is the Lagrangean sound speed; note that their signs are fixed. It follows immediately that

$$\tilde{A} = A^2/C, \tag{26}$$

hence

$$q_0 = q_1 = 0, \qquad q_2 = 1/(\lambda C). \tag{27}$$

Note the similarity to scheme (16), for which

$$q_0 = q_1 = 0, \qquad q_2 = 1, \tag{28}$$

irrespective of the equations.

For the Euler equations the situation is more complicated, since the characteristic speeds do not have fixed signs. They are $u - c$, u and $u + c$, where c is the usual sound speed and u is the flow speed. There are four different combinations of signs. For supersonic flow (all signs equal) we immediately find that

$$\tilde{A} = A \operatorname{sgn} u, \tag{29}$$

while for zero flow speed we must get back the Lagrangean result (26), with C replaced by c. Written in terms of the Mach number

$$M \equiv u/c, \tag{30}$$

the complete formulas for the matrix (18) are

$$\begin{aligned} &q_0 = q_2 = 0, \qquad q_1 = \operatorname{sgn} M, \qquad \text{for } |M| \geq 1, \\ &\left.\begin{aligned} q_0 &= \lambda c\,|M|\,(1 - M^2), \\ q_1 &= M(2\,|M| - 1), \\ q_2 &= (1 - |M|)/(\lambda c), \end{aligned}\right\} \quad \text{for } |M| < 1. \end{aligned} \tag{31}$$

In Fig. 2 these coefficients are plotted against M. The switch is buried so deep inside scheme (15) that, for $|M| < 1$, the upstream centering is totally obscured.

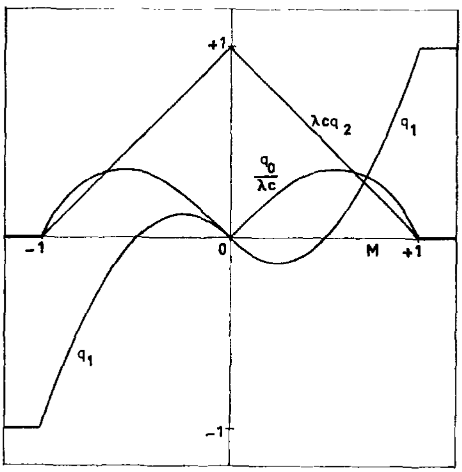

FIG. 2. Dependence of the coefficients in the polynomial for $\lambda\tilde{A}$ on the Mach number in the Eulerian case.

With the above coefficients, the two-step Euler version of scheme (15) becomes a bit "heavy" for a first-order scheme. Commonly, first-order schemes are one-step Q schemes with

$$Q = q_0 I, \tag{32}$$

the computational ease of which is obvious. The possibilities implied in (32) are discussed by Van Leer [16]. One particular example is the scheme of Rusanov [9], in which q_0 contains an adjustable, position-independent factor. The best results are generally obtained with Q equal to or close to

$$Q_{\text{Rusanov}} = \lambda \max_k |a^{(k)}|\, I. \tag{33}$$

When comparing (33) with (18) we find that Rusanov's scheme treats the underlying convection equations less accurately than Godunov's, except for the one with the highest convection speed.

The chief drawback of Rusanov's scheme is that it yields diffusion of entropy even for zero flow speeds. This is because, if the eigenvalue u of A vanishes, the corresponding eigenvalue of Q does not vanish (see, e.g., [11, Sec. 5]). The same holds for all other schemes employing (32). It appears now that Rusanov [20] actually succeeded in removing the residual diffusion from his scheme through a slight change in the $Q\,\Delta w$ terms, but thought it needless to do the same with the two-dimensional version [9] that became so popular. Godunov's scheme does not suffer from the above defect, but neither does the simpler LW scheme. The latter, again, is less satisfying in shock problems, witness Fig. 1.

I do not intend here to weigh all the pros and cons of the different Q schemes. Suffice it to say that the two-step Godunov scheme combines some desirable properties, among which, however, is *not* computational ease. Bearing this in mind we may expect the worst for the more complicated higher-order upstream schemes, to which I shall turn now.

3. Higher-order Schemes

Fromm's scheme, when applied to Eq. (6), reads

$$w^0 = \begin{cases} w_0 - \sigma\,\Delta_{-1/2}w - (\sigma/4)(1-\sigma)(\Delta_{1/2} - \Delta_{-3/2})\,w & \text{if } \sigma \geqslant 0, \\ w_0 - \sigma\,\Delta_{1/2}w + (\sigma/4)(1+\sigma)(\Delta_{3/2} - \Delta_{-1/2})\,w & \text{if } \sigma < 0. \end{cases} \tag{34}$$

It is the simplest member of the family of second-order upstream schemes

$$w^0 = \begin{cases} w^0 - \sigma\,\Delta_{-1/2}w - (\sigma/4)(1-\sigma)(\Delta_{1/2} - \Delta_{-3/2})\,w \\ \qquad - (\varphi/4)(\Delta_{1/2} - 2\Delta_{-1/2} + \Delta_{-3/2})\,w & \text{if } \sigma \geqslant 0, \\ w^0 - \sigma\,\Delta_{1/2}w + (\sigma/4)(1+\sigma)(\Delta_{3/2} - \Delta_{-1/2})\,w \\ \qquad + (\varphi/4)(\Delta_{3/2} - 2\Delta_{1/2} + \Delta_{-1/2})\,w & \text{if } \sigma < 0, \end{cases} \tag{35}$$

where φ is an even function of σ and an uneven function of $|\sigma| - \frac{1}{2}$. A Fourier analysis of (35) reveals that, within the range of the CFL condition (7), the scheme is dissipative provided

$$-1 + |\sigma| \leqslant \varphi \leqslant |\sigma|. \tag{36}$$

The dispersive and dissipative errors turn out to be particularly small if we choose

$$\varphi = \kappa\,|\sigma|\,(1 - |\sigma|)(1 - 2\,|\sigma|), \tag{37}$$

where κ is a constant. Condition (36) then reduces to

$$-8 \leqslant \kappa \leqslant 1. \tag{38}$$

For $\kappa = \frac{1}{3}$, third-order accuracy is achieved.

In converting (34) into a conservative scheme for Eq. (1), we may start with the procedure defined in Eqs. (11) and (12). This yields

$$\begin{aligned} w^0 = {} & w_0 - \sigma\, \Delta_0 w + (\sigma^2/2)(\Delta_{1/2} - \Delta_{-1/2})\, w \\ & + (\sigma/8)(1 - |\sigma|)(\Delta_{3/2} - \Delta_{1/2} - \Delta_{-1/2} + \Delta_{-3/2})\, w \\ & - (|\sigma|/8)(1 - |\sigma|)(\Delta_{3/2} - 3\Delta_{1/2} + 3\Delta_{-1/2} - \Delta_{-3/2})\, w. \end{aligned} \tag{39}$$

The intended conservative scheme is readily obtained if we replace w by w, σ by λA, $\sigma\, \Delta w$ by $\lambda\, \Delta f$, and $|\sigma|$ by $\lambda \tilde{A}$, thereby centering the matrix coefficients on the same meshes as the finite differences that they multiply. The result looks like a LW scheme, augmented by terms of the third and the fourth order. These do not affect the order of consistency of the scheme; they just bring about the upstream centering.

A two-step version of the scheme is warranted in view of the appearance of the matrix coefficients $\lambda\tilde{A}$, $\lambda^2 A^2$ and $\lambda^2 \tilde{A} A$. The latter one is a newcomer; in the Lagrangean case it offers no problems, since

$$\tilde{A} A = CA. \tag{40}$$

In the Eulerian case, expanding $\lambda^2 \tilde{A} A$ in the manner of Eq. (24) yields

$$\begin{aligned} & q_0 = q_1 = 0, \qquad q_2 = \operatorname{sgn} M, \qquad \text{if} \quad |M| \geqslant 1, \\ & \left.\begin{aligned} q_0 &= -\lambda^2 c^2 M(1 + |M|)(1 - |M|)^2, \\ q_1 &= \lambda c(1 + 2\,|M|)(1 - |M|)^2, \\ q_2 &= M(2 - |M|), \end{aligned}\right\} \qquad \text{if} \quad |M| < 1. \end{aligned} \tag{41}$$

Without spelling out its two-step version it is readily understood that the Fromm-based conservative scheme involves many more operations than the two-step LW scheme. Even for the Lagrangean equations programming is awkward, because of the many nontrivial terms.

In order to judge the economy of the above upstream scheme, one needs to compare it in practice with central-difference schemes. So far, numerical comparisons have been made only on the basis of a single linear convection equation, thus avoiding the switching problem inherent in upstream schemes. Such tests may tell us something about the relative accuracy of the schemes investigated, but not about the relative efficiency of the schemes when applied to a system of nonlinear conservation laws.

Fortunately, one clue is offered by Wesseling [5], who derived and/or tested various schemes of the upstream family (35), among which are Fromm's scheme and the third-order scheme. His tests also include several central-difference schemes, among which are the LW scheme and the third-order (RBM) scheme derived by Rusanov [17] and also studied by Burstein and Mirin [18]. The RBM scheme is a three-step scheme for Eq. (1) with several parameters; its simplest form is

$$\begin{aligned} w_{1/2}^{*} &= w_{1/2} - (\lambda/3)\, \Delta_{1/2} f, \\ w_0^{**} &= w_0 - (2\lambda/3)\, \Delta_0 f^{*}, \\ w^0 &= w_0 - (\lambda/24)(-2f_2 + 7f_1 - 7f_{-1} + 2f_{-2}) \\ &\quad - (3\lambda/4)\, \Delta_0 f^{**} - (1/24)\{\Delta^3(\omega\, \Delta w)\}_0 . \end{aligned} \tag{42}$$

The choice of the scalar ω is restricted by the stability condition

$$\lambda^2 A^2(4I - \lambda^2 A^2) \leqslant \omega I \leqslant 3I. \tag{43}$$

Wesseling finds the accuracy of the RBM scheme "disappointing" unless ω is replaced by a matrix Ω, in particular, by the lower bound in Eq. (43). With

$$\Omega_{\min} = \lambda^2 A^2(4I - \lambda^2 A^2), \tag{44}$$

the scheme is fourth-order accurate in space, although still third-order accurate in time.

With this form of the RBM scheme in mind, Wesseling suggests that "the most economical way to increase the accuracy [of flow computations] may not be to use the RBM scheme, but to develop a predictor–corrector form of one of the 5-point schemes [that is, schemes of the family (35)], and use this with slightly diminished step-size." This conjecture now is easy to verify. For, once ω is upgraded to a matrix, the general RBM scheme (42) also includes the third-order *upstream* scheme. Using the procedure defined in Eqs. (11) and (12) we find it has the following value of Ω:

$$\Omega_{\text{upstream}} = \lambda\tilde{A}(I + 2\lambda\tilde{A})(2I - \lambda\tilde{A}). \tag{45}$$

Wesseling suggests explicit calculation of the matrix Ω and the product $\Omega\Delta w$, which would make the scheme with (45) hardly more expensive than the scheme with (44).

It is tempting to expand Ω in the usual way:

$$\Omega = \omega_0 I + \omega_1 \lambda A + \omega_2 \lambda^2 A^2, \tag{46}$$

and then account for the different powers of λA in the various steps of the RBM scheme. We may, for instance, include the whole expansion in the last step of the RBM scheme, by changing the original ω term into

$$-\tfrac{1}{24}[\{\Delta^3(\omega_0 \Delta w)\}_0 + \lambda\{\Delta^3(\omega_1 \Delta f)\}_0 - 3\lambda\{\Delta^3[\omega_2(f^* - f)]\}_0]. \tag{47}$$

Again, the two Ω schemes will not differ much in computing time.

However, there is no need to expand $\Omega_{\min}$ once we adopt a *four*-step formulation. This matrix can be built up in four successive steps using constant coefficients only; fourth-order accuracy in time may even be achieved in the process. For Ω_{upstream} this does not work: no matter what powers of λA we use in the expansion, we shall always end up with coefficients that depend on the eigenvalues of A, such as given in Eqs. (31) and (41). Hence, the upstream scheme ultimately is the least economic of the two Ω schemes.

The results of this section may be summarized as follows: *In a conventional predictor-corrector formulation, upstream-difference schemes seem to be defeated by central-difference schemes of a higher order of consistency.* In arriving at this conclusion I have left out of consideration the measures to prevent nonlinear instabilities (see [15]) and the related measures to prevent numerical oscillations (see [19]). These add to the execution time of both central and upstream schemes. I do not know whether this will result in drawing the efficiencies of the two kinds of schemes closer together or further apart. So far, the method of preventing oscillations given in [19] has been applied only to Fromm's scheme.

In [19] the extra terms added to Fromm's scheme (34) for the sake of monotonicity were cast into the form of the φ terms occurring in Eq. (35). They are feedback terms, designed to neutralize the second-order terms of the scheme in all cases where the latter would give rise to numerical oscillations. In order to achieve this automatically, φ is made a function of the "smoothness monitor"

$$\vartheta_i \equiv \Delta_{i+1/2}w/\Delta_{i-1/2}w, \tag{48}$$

a quantity that measures the local rate of change of $\partial w/\partial x$ across a mesh. In a sufficiently fine grid the value of ϑ will nowhere differ much from unity, except in the neighborhood of a discontinuity in w or $\partial w/\partial x$. Correspondingly, φ may be chosen to vanish everywhere except in those few danger areas.

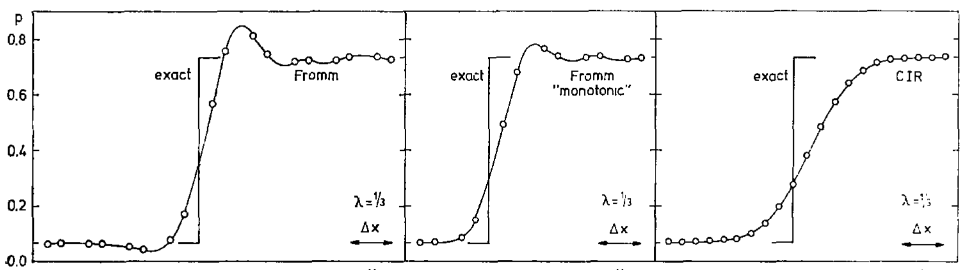

FIG. 3. Numerical pressure profiles of a shock obtained with various upstream conservative schemes for the Lagrangean flow equations. The postshock Courant number equals 0.6383 in all cases. Left: results of two-step Lagrangean version of Fromm's scheme. Center: same as left but using the nonlinear damping technique of [19] to suppress numerical oscillations. Smoothness is monitored solely by the rate of change of the Lagrangean sound speed C; the coefficient of the damping terms is given by [19, Eq. 29]. Right: results of Lagrangean version of CIR scheme.

When the full Fromm scheme is converted into a conservative scheme for Eq. (1), the extra terms transform as usual. Note that, in principle, each characteristic state quantity $w^{(k)}$ (see Eq. (5)) requires its own smoothness monitor! In practice, however, monitoring just one sensitive quantity, such as the pressure, will suffice.

My experience with the feedback terms is limited to the Lagrangean case and is somewhat disappointing: they are not foolproof. One reason is that, in the two-step formulation, there are so many of them that their effect becomes difficult to control. An example of their influence on a numerical shock profile is given in Fig. 3.

In the next two installments of the present series I shall adopt and extend Godunov's approach to the formulation of upstream conservative schemes. This approach involves more of the physics implied by the conservation laws than just the matrix A and its eigenvalues. The resulting schemes have a number of advantages over the ones presented here, central-difference schemes included. In particular, they are transparent and can easily be made monotonic. Also, they can be made more accurate than the corresponding finite-difference schemes. Therefore, the case of the higher-order upstream schemes for conservation laws remains open.

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